

Graded geometry in gauge theories and beyond

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Abstract

We study some graded geometric constructions appearing naturally in the context of gauge theories. Inspired by a known relation of gauging with equivariant cohomology we generalize the latter notion to the case of arbitrary Q -manifolds introducing thus the concept of equivariant Q -cohomology. Using this concept we describe a procedure for analysis of gauge symmetries of given functionals as well as for constructing functionals (sigma models) invariant under an action of some gauge group.

As the main example of application of these constructions we consider the twisted Poisson sigma model. We obtain it by a gauging-type procedure of the action of an essentially infinite dimensional group and describe its symmetries in terms of classical differential geometry.

We comment on other possible applications of the described concept including the analysis of supersymmetric gauge theories and higher structures.

Keywords: Q -manifolds, Equivariant cohomology, Gauging, Twisted Poisson sigma model, Courant algebroids.

2010 MSC: 58A50, 53D17, 70S15, 55N91

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1. Introduction/preliminaries

In this paper we describe the possibilities offered by graded geometry for the analysis of physical theories and some objects of classical differential geometry. We introduce a powerful tool – the concept of equivariant Q -cohomology which is a natural extension of the definition of standard equivariant cohomology to Q -manifolds.

In the first part of the paper after introducing the problem of gauging we briefly sketch some facts from the theory of Q -manifolds and fix the notations in the examples that are important in what follows. In the section 2 we define the notion of equivariant Q -cohomology and explain, following the scheme of A. Kotov and T. Strobl ([1], [2]), its relation to gauge invariance. Within this framework we recover explicitly the result of J.M. Figueroa-O'Farrill and S. Stanciu [3] on the obstruction to gauging of the Wess-Zumino terms – this shows also how ordinary equivariant cohomology can be obtained as a particular case of Q -cohomology. The section 3 is entirely devoted to the analysis of the twisted Poisson sigma model ([4]): we describe the algebra of its symmetries and construct its functional using the procedure that we suggest as an alternative to standard gauging. In the last section 4 we give a purely mathematical application of the described concept, namely we propose a possible definition of equivariant cohomology for Courant algebroids. To conclude we also comment on other applications and some work in progress.

1.1. Gauging problem, Wess-Zumino terms

The major part of this work is motivated by the gauging problem, which is important in theoretical physics. To give a simple example of the procedure consider $X: \Sigma^d \rightarrow M^n$ – a map between two smooth manifolds of dimensions d and n respectively, and $B \in \Omega^d(M)$; in the physicist's terminology one would call X a scalar field, Σ – world-sheet and M – target. Assume that a Lie group G acts on M and leaves B invariant. This induces a G -action on M^Σ , which leaves invariant the functional $S[X] = \int_\Sigma X^*B$. The invariance with respect to G is called a (global) *rigid invariance*. The functional is called (locally) *gauge invariant*, if it is invariant with respect to the group $G^\Sigma \equiv C^\infty(\Sigma, G)$; it is clear that in its original form $S[X]$ is not necessarily gauge invariant.

The procedure of gauging consists in modifying the functional S in order to make it gauge invariant. This is usually done by introducing new variables to S controlling however that the result reduces to the initial functional

when the additional variables are put to zero – these new variables may have some physical meaning in concrete applications. For the above functional the gauging problem can be solved by extending S to a functional \tilde{S} defined on $(X, A) \in M^\Sigma \times \Omega^1(\Sigma, \mathfrak{g})$, $\mathfrak{g} = \text{Lie}(G)$ by means of so-called minimal coupling. For example if $d = 2$ the result is

$$\tilde{S}[X, A] = \int_{\Sigma} \left(X^* B - A^a X^* \iota_{v_a} B + \frac{1}{2} A^a A^b X^* \iota_{v_a} \iota_{v_b} B \right),$$

where $A^a, A^b, \iota_{v_a}, \iota_{v_b}$ are defined by fixing the basis of \mathfrak{g} .¹ Making again a remark about the physicist's terminology one would call the variables A the one-form valued (gauge) fields.

Suppose now that $\Sigma^d = \partial\Sigma^{d+1}$, $\tilde{X}: \Sigma^{d+1} \rightarrow M$ – an extension of X coinciding with it on Σ^d . Let $H \in \Omega^{d+1}(M)$ be a closed differential form invariant under the induced action of G . The functional $S[X] = \int_{\Sigma^{d+1}} \tilde{X}^* H$ is the simplest one containing the so-called *Wess-Zumino term* (the integration is performed over the bulk of the worldsheet manifold, [5]). In contrast to the previous example gauging of this functional can be obstructed. More precisely in [3, 6] it has been shown that gauging is possible, if and only if H permits an equivariantly closed extension. There is however a serious limitation for the procedure suggested in [3], namely the number of introduced gauge fields is equal to the dimension of the group G making it not very practical for essentially infinite dimensional groups which do appear in applications. In what follows we will see how this issue can be treated within the framework of Q -bundles and in particular observe the situations when the extension of the target is not given explicitly by the Lie algebra of the group acting.

1.2. Graded geometry, Q -manifolds

We are certainly not going to give a full introduction to graded geometry, referring to nice sources like [7, 8, 9, 10]. Let us however give a definition of a Q -manifold and several examples of it.

Definition 1.1. *A Q -manifold is graded manifold equipped with a Q -structure – a degree 1 vector field Q satisfying $[Q, Q] \equiv 2Q^2 = 0$.*

¹Here and in the whole text the convention of summation over repeating indices is adopted.

Example 1.1. Consider any smooth manifold M , declare fiber linear coordinates of the tangent bundle to it to be of degree 1. The graded manifold obtained like this is generally denoted $T[1]M$. It is equipped with the de Rham differential that written in local coordinates $(x^i, \theta^i = dx^i)$ has the form $d_{dR} = \theta^i \frac{\partial}{\partial x^i}$, it thus can be viewed as a degree 1 vector field squaring to zero.

Example 1.2. Consider a Lie algebra \mathfrak{g} , choose a basis of it and declare local coordinates ξ^a to be of degree 1. This graded manifold denoted $\mathfrak{g}[1]$ is equipped with the Chevalley-Eilenberg differential $Q_{CE} = C_{bc}^a \xi^b \xi^c \frac{\partial}{\partial \xi^a}$, where C_{bc}^a are the structure constants of \mathfrak{g} . $Q_{CE}^2 = 0$ is equivalent to the Jacobi identity.

A more involved example is provided by a twisted Poisson manifold M . Let us recall that given a closed differential form $H \in \Omega^3(M)$ an almost Poisson bivector Π is called *twisted Poisson* if and only if it satisfies the twisted version of the Jacobi identity: $[\Pi, \Pi]_{SN} = (\Pi^\#)^{\otimes 3}(H)$, where $[\cdot; \cdot]_{SN}$ is the Schouten-Nijenhuis bracket of multivector fields and the right hand side of the equality denotes the full contraction of H with Π . In this case the couple (Π, H) is called a *twisted Poisson structure*.

Example 1.3. Consider a cotangent bundle to a manifold M equipped with a twisted Poisson structure (Π, H) . A graded manifold obtained by shifting the grading of the fiber linear coordinates p_i by 1 is usually denoted $T^*[1]M$. For $C_i^{jk}(x) = \frac{\partial \pi^{jk}}{\partial x^i} + H_{ij'k'} \pi^{jj'} \pi^{kk'}$ consider the degree 1 vector field

$$Q_{\pi, H} = \pi^{ij} p_j \frac{\partial}{\partial x^i} - \frac{1}{2} C_i^{jk} p_j p_k \frac{\partial}{\partial p_i}.$$

$Q_{\pi, H}^2 = 0$ is precisely equivalent to the twisted Jacobi identity. Note that in the “untwisted” case of $H = 0$ this vector field is hamiltonian with respect to the canonical Poisson structure on $T^*[1]M$: $Q_{\pi, 0} = \{\frac{1}{2} \pi^{ij} p_i p_j, \cdot\}$.

This is certainly a non-exhaustive list of examples: the objects like Dirac structures, Lie algebroids (cf. [11]), Courant algebroids also can fit to the picture. We will discuss the latter one in more details in section 4.

Having defined the Q -structure on a graded manifold one has also a natural definition of a morphism:

Definition 1.2. *Given two Q -manifolds (\mathcal{M}_1, Q_1) , (\mathcal{M}_2, Q_2) , a degree preserving map $f: \mathcal{M}_1 \rightarrow \mathcal{M}_2$, is a Q -morphism if and only if on all superfunctions it commutes with the action of the respective vector fields: $Q_1 f^* - f^* Q_2 = 0$.*

A generic degree preserving map φ fails to be a Q -morphism but there is always a canonical construction permitting to extend φ to a Q -morphism. More precisely the following statement holds true.

Proposition 1.1. *Given a degree preserving map φ between Q -manifolds (\mathcal{M}_1, Q_1) and (\mathcal{M}_2, Q_2) , there exists a Q -morphism f between the Q -manifolds (\mathcal{M}_1, Q_1) and $(\tilde{\mathcal{M}}, \tilde{Q}) = (T[1]\mathcal{M}, d_{DR} + \mathcal{L}_{Q_2})$, covering φ .*

Proof. Fix a coordinate system q^a on \mathcal{M}_2 , the vector field Q_2 reads $Q^a \frac{\partial}{\partial q^a}$. From the Cartan's magic formula one immediately sees that $\tilde{Q}q^a = dq^a + Q^a$. Since we want f to cover φ , f^* is forced to coincide with φ on all the functions of q . To extend it to a Q -morphism on the whole $\tilde{\mathcal{M}}$ it is sufficient to define $f^*(dq^a) = Q_1 f^*(q^a) - f^*(Q^a)$ and note that \tilde{Q} commutes with d . For details and a more geometric interpretation, see the discussion on the “field strength” in [1] as well as the proposition 3.3 there. \square

Remark 1.1. *It is important to note that in this construction there is a double grading appearing naturally: each homogeneous function on the resulting graded manifold $\tilde{\mathcal{M}}$ can inherit a degree from \mathcal{M}_2 or from the shift in the tangent bundle. We will use the Bernstein–Leites sign convention for treating this double grading, i.e. the sign in the commutation relations is governed by the sum of the degrees of each element.*

2. Equivariant Q -cohomology

We have mentioned in the previous section that the problem of gauging of the Wess-Zumino terms is related to equivariant extensions. Just to motivate the coming definitions let us note that in the ordinary case main objects for computing equivariant cohomology are the contractions of differential forms with vector fields induced by the group action and the de Rham differential (for details see for example [12, 13]). In the language of graded manifolds the contraction decreases the degree of any differential form (viewed as a superfunction) and the de Rham differential increases. Let us now generalize this construction to arbitrary Q -manifolds.

Let (\mathcal{M}, Q) be a Q -manifold, and let \mathcal{G} be a subalgebra of degree -1 vector fields ε on \mathcal{M} , closed with respect to the Q -derived bracket $([\varepsilon, \varepsilon']_Q \equiv [\varepsilon, [Q, \varepsilon']])$. We consider the action of Q on superfunctions on \mathcal{M} as a *generalized differential* and the action of any ε as a *generalized contraction*.

Definition 2.1. A superfunction ω on \mathcal{M} is \mathcal{G} -horizontal if and only if $\varepsilon\omega = 0, \forall \varepsilon \in \mathcal{G}$.

Definition 2.2. A superfunction ω on \mathcal{M} is \mathcal{G} -equivariant if and only if $(ad_Q \varepsilon)\omega \equiv [Q, \varepsilon]\omega = 0, \forall \varepsilon \in \mathcal{G}$.

Definition 2.3. We call a superfunction ω on \mathcal{M} \mathcal{G} -basic if and only if it is \mathcal{G} -horizontal and \mathcal{G} -equivariant.

Remark 2.1. Like in the standard case within the framework of the above definitions one can consider a family of equivariant differentials $d_\varepsilon = Q + \varepsilon$, each of them does not square to zero by itself, but does on \mathcal{G} -basic superfunctions.

Remark 2.2. It is easy to see that for Q -closed superfunctions being \mathcal{G} -horizontal is equivalent to being \mathcal{G} -basic.

Remark 2.3. The Q -derived bracket need not be skew-symmetric, it defines thus a Loday- but in general not a Lie algebra structure. However, the definitions above are also valid for the particular case when \mathcal{G} is isomorphic to a Lie algebra $\mathfrak{g} = Lie(G)$, like in the classical definition of equivariant cohomology. In this case the restriction of the Q -derived bracket to the considered vector fields is necessarily skew-symmetric.

2.1. Gauge invariance and Q -bundles

The reason for considering these objects is that they appear naturally in the description of sigma models via Q -bundles proposed by A. Kotov and T. Strobl ([1]). Let us sketch the approach here.

Associate Q -manifolds (\mathcal{M}_1, Q_1) and (\mathcal{M}_2, Q_2) respectively to the world-sheet and the target of a gauge theory and encode the fields in a degree preserving map φ between them. As we have seen in the proposition 1.1, one can lift φ to a Q -morphism. One should actually extend the target even more by considering the direct product $(\hat{\mathcal{M}}, \hat{Q}) = (\mathcal{M}_1 \times \tilde{\mathcal{M}}, Q_1 + \tilde{Q})$ and the appropriate extension of φ and f , this gives a Q -bundle – see the diagram

(1) where by abuse of notation we call the extended maps by the same letters φ and f .

$$\begin{array}{ccc}
 & (\mathcal{M}_1 \times \tilde{\mathcal{M}}) \circ \hat{Q} & \\
 & \nearrow f & \\
 Q_1 \circ \mathcal{M}_1 & \xrightarrow{\varphi} & \mathcal{M}_1 \times \mathcal{M}_2 \\
 & \nwarrow pr_1 & \uparrow
 \end{array} \tag{1}$$

The key idea is that within the framework of this construction the gauge transformations can be parametrized by $\hat{\varepsilon}$ – vector fields on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$ of total degree -1 , vertical with respect to the projection pr_1 to \mathcal{M}_1 :

$$\delta_\varepsilon(f^*\cdot) = f^*([\hat{Q}, \hat{\varepsilon}]\cdot). \tag{2}$$

One can consider separately the dependence of $\hat{\varepsilon}$ on \mathcal{M}_1 and $\tilde{\mathcal{M}}$, and thus construct the algebra \mathcal{G} of degree -1 vector fields on $\tilde{\mathcal{M}}$. For the functionals of the form $S = \int_{\Sigma^{d+1}} f^*(\omega)$ gauge invariance would be guaranteed precisely by the condition of ω being \mathcal{G} -basic (definition 2.3) and \tilde{Q} -closed (cf. also the remark 2.2).

Remark 2.4. *A natural question to ask is how generic is the situation when one can proceed with the above construction. For the world-sheet manifold usually there is no problem: one often considers $T[1]\Sigma$ as the Q -manifold (cf. example 1.1). For the target according to [14] the Q -structure exists when field equations satisfy a certain type of Bianchi identities, which is not a very restrictive condition.*

2.2. Gauging via equivariant Q -cohomology

In view of the previous subsection it is natural to consider the integrand of the functional with the rigid symmetry group in the form of a pull-back by a Q -morphism from the target manifold of some superfunction ω . The gauging problem reduces to finding a \mathcal{G} -basic \tilde{Q} -closed extension $\tilde{\omega}$ of this superfunction. As we have understood, the necessary data for this procedure is the geometry of the target encoded in the Q -structure and the morphism from the algebra of symmetries to the algebra \mathcal{G} of degree -1 vector fields with a \tilde{Q} -derived bracket. The rest is a straightforward application of the conditions coming from the definition 2.3:

$$\tilde{Q}\tilde{\omega} = 0, \quad \tilde{\varepsilon}\tilde{\omega} = 0, \quad \forall \tilde{\varepsilon} \in \mathcal{G}. \tag{3}$$

As an example of this procedure let us consider gauging of the Wess-Zumino term for $\dim \Sigma = 2$ and the extension of the target being governed by a Lie group G acting on M . An appropriate geometric structure for this case is the action Lie algebroid $E = M \times \mathfrak{g}$. Declaring like in the example 1.2 the coordinates on \mathfrak{g} to be of degree 1 we obtain the graded manifold usually denoted as $E[1]$ with a Q -structure $Q_g = \rho - Q_{CE}$, where $\rho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ is the action of \mathfrak{g} on M . In some local chart $Q_g = \xi^a \rho_a^i \frac{\partial}{\partial x^i} - C_{bc}^a \xi^b \xi^c \frac{\partial}{\partial \xi^a}$. As explained above, we consider the target Q -manifold $(\tilde{\mathcal{M}}, \tilde{Q}) = (T[1]E[1], d + \mathcal{L}_{Q_g})$. For the symmetry algebra the construction is also rather straightforward: we consider the degree -1 vector fields on $E[1]$ that are all of the form $\varepsilon = \varepsilon^a \frac{\partial}{\partial \xi^a}$ and lift them to $T[1]E[1]$ by a Lie derivative $\tilde{\varepsilon} = \mathcal{L}_\varepsilon$. One can check by a straightforward computation that the \tilde{Q} -derived bracket of two such vector fields is governed by a Lie-algebra bracket of the respective elements of \mathfrak{g} . A 3-form H on M can be viewed as a superfunction on $T[1]E[1]$, the problem of extending the Wess-Zumino term defined by it can thus be formulated in the Q -language presented above.

To solve the extension problem let us apply the conditions (3) to the most general degree superfunction \tilde{H} on $\tilde{\mathcal{M}}$ writing it in local coordinates. It is convenient to choose the basis generated by $x^i, \xi^a, \tilde{Q}x^i, \tilde{Q}\xi^a$ of total degrees 0, 1, 1 and 2 respectively. The first condition of \tilde{Q} -closedness reduces the extension to $\tilde{H} = \frac{1}{6}H_{ijk}\tilde{Q}x^i\tilde{Q}x^j\tilde{Q}x^k + E_{ia,j}\tilde{Q}x^i\tilde{Q}x^j\xi^a + \frac{1}{2}F_{ab,i}\tilde{Q}x^i\xi^a\xi^b + E_{ia}\tilde{Q}x^i\tilde{Q}\xi^a + F_{ab}\xi^a\tilde{Q}\xi^b$, subject to $F_{(ab)} = 0$.² The second condition amounts to several equalities, among which one has:

$$\frac{1}{2}H_{ijk}\varepsilon^a\rho_a^i + (E_{j]a}\varepsilon^a)_{, [k} = 0, \quad (E_{ia}\rho_b^i + F_{ba})\varepsilon^b.$$

They restrict the extension to

$$\tilde{H} = \frac{1}{6}H_{ijk}\tilde{Q}x^i\tilde{Q}x^j\tilde{Q}x^k + \tilde{Q}(\frac{1}{2}E_{ja}\rho_b^j\xi^a\xi^b - E_{ia}\tilde{Q}x^i\xi^a),$$

that repeats the solution given in [3].

Remembering about the antisymmetry of $F_{ab} = E_{ia}\rho_b^i$ one recovers the same obstructions to gauging as in [3], that can be interpreted as the existence of the equivariantly closed extension of H (in terms of standard equivariant

²Here and in the whole text $\cdot_{,i}$ denotes the derivative w.r.t. x^i ; $(\cdot\cdot)$ – symmetrized and $[\cdot\cdot]$ antisymmetrized indices

de Rham theory). This observation permits to validate the definitions 2.1 - 2.3 in the sense that they indeed generalize the standard picture.

Remark 2.5. *Some similar results about equivariant Q -cohomology are discussed in [15], where it is shown that, when the action is free and proper, then it agrees with the Q -cohomology of the quotient.*

3. Twisted Poisson sigma model

As we have already mentioned, the approach of the previous section is not limited to the target extended by the group to be gauged. In this section we will describe in details the application of it to analysis of the twisted Poisson sigma model.

As in the previous example the world-sheet is given by a smooth manifold Σ^2 (closed, orientable, with no boundary, $\dim = 2$). The target is a smooth manifold M^n with a (twisted) Poisson structure (Π, H) (cf. example 1.3). The functional over the space of vector bundle morphisms $T\Sigma \rightarrow T^*M$ reads

$$S[X, A] = \int_{\Sigma} A_i \wedge dX^i + \frac{1}{2} \pi^{ij} A_i \wedge A_j + \int_{\Sigma^3} H, \quad (4)$$

where $X^i : \Sigma \rightarrow M$ are scalar fields and $A_i \in \Omega^1(\Sigma, X^*T^*M)$ – 1-form valued (“vector”) fields. For details about the definition, motivation and physical significance of the model one can refer to original sources: [16], [4], as well as [17], [18], [19]. One should also note that the model is related to derivation of the famous Kontsevitch’s formula ([20]) for quantization of Poisson manifolds.

3.1. Symmetries of twisted PSM

As it is suggested by the examples in the beginning of the section 1.2 the natural Q -manifolds to use as worldsheet and target are $(T[1]\Sigma, d_{dR})$ and $(T^*[1]M, Q_{\Pi, H})$. To recover the Q -morphism from the proposition 1.1 we lift the picture to $(\tilde{\mathcal{M}}, \tilde{Q}) = (T[1]T^*[1]M, d + \mathcal{L}_{Q_{\Pi, H}})$. The most general degree -1 vector field on $T^*[1]M$ reads $\varepsilon = \varepsilon_i \frac{\partial}{\partial p_i}$. To get into the framework of the section 2.1 we lift this vector field to $\tilde{\mathcal{M}}$ by the Lie derivative. The \tilde{Q} -derived bracket of such vector fields induces on sections of T^*M a Lie algebra structure (twisted Poisson Lie algebroid bracket):

$$[\varepsilon^1, \varepsilon^2] = \mathcal{L}_{\pi^{\#}\varepsilon^1}\varepsilon^2 - \mathcal{L}_{\pi^{\#}\varepsilon^2}\varepsilon^1 - d(\pi(\varepsilon^1, \varepsilon^2)) + \iota_{\pi^{\#}\varepsilon^1}\iota_{\pi^{\#}\varepsilon^2}H, \quad (5)$$

defining thus a Lie algebra that we denote \mathcal{G} . With this natural geometric construction we can already formulate a statement about gauge symmetries of the twisted PSM.

Theorem 3.1. *Any smooth map from Σ to the space $\Gamma(T^*M)$ of sections of the cotangent bundle to M defines an infinitesimal gauge transformation of the twisted PSM in the sense of equation (2) if and only if for any point $\sigma \in \Sigma$ the section ε satisfies*

$$d\varepsilon + \iota_{\pi^\# \varepsilon} H = 0, \quad (6)$$

where d is the de Rham differential on M .

Proof (Sketch). To prove the statement one needs to rewrite the functional of the twisted PSM in the form $S = \int_{\Sigma_3} f^*(\tilde{H})$ and apply to \tilde{H} the conditions (3). The first one will be satisfied automatically since H is a closed form, the second one will reduce to the equation (6). We will give an explicit formula in the proof of the theorem 3.2. \square

Remark 3.1. *This theorem permits to construct all the essential gauge transformations of the twisted PSM (a generating set), we will however return to the question of enlarging the algebra of symmetries. The theorem specializes to the ordinary “untwisted” PSM by setting $H = 0$ and one recovers the (properly understood) condition on ε from [21].*

If we consider the result of application of the equivariant Q -cohomology procedure for the subalgebra of \mathcal{G} defined by (6), it can be ambiguous. The way out is to consider the full (modulo some technical assumptions) algebra $\tilde{\mathcal{G}}$ of degree -1 vector fields on $\tilde{\mathcal{M}}$. To define $\tilde{\mathcal{G}}$ consider $\tilde{\varepsilon}$ – a degree -1 vector field on $\tilde{\mathcal{M}}$ of the form

$$\tilde{\varepsilon} = \varepsilon_i \frac{\partial}{\partial p_i} + \alpha_{ij} \theta^j \frac{\partial}{\partial \psi_i} \quad (7)$$

Remark 3.2. *Let us note, that the vector field $\tilde{\varepsilon}$ introduced above is not of the most general form that one could have on $\tilde{\mathcal{M}}$: compare it with*

$$\tilde{\varepsilon} = \varepsilon_i \frac{\partial}{\partial p_i} + \bar{\varepsilon}^i \frac{\partial}{\partial \theta^i} + (\alpha_{ij} \theta^j + \bar{\alpha}_i^j p_j) \frac{\partial}{\partial \psi_i}$$

But the Lie bracket of two such vector fields would in general produce a degree -2 vector field, namely

$$[\tilde{\varepsilon}, \tilde{\varepsilon}'] = (\bar{\alpha}'^j_i \varepsilon_j - \bar{\alpha}^j_i \varepsilon'_j + \alpha'_{ij} \tilde{\varepsilon}^j - \alpha_{ij} \tilde{\varepsilon}'^j) \frac{\partial}{\partial \psi_i}$$

So, to stay inside the set of vector fields of degree precisely -1 and generalize $\tilde{\varepsilon} = \mathcal{L}_\varepsilon$ we have to keep only the “unbarred” components.

It turns out, that this Lie algebra is rather natural, i.e. one can describe it up to isomorphism using classical differential geometry. More precisely the following proposition holds:

Proposition 3.1. *The Lie algebra $(\tilde{\mathcal{G}}, [,]_{\tilde{Q}})$ is isomorphic to the semi-direct product of Lie algebras $\mathcal{G} \in \mathcal{A}$, where \mathcal{A} is a Lie algebra of covariant 2-tensors on M with a bracket given by*

$$[\bar{\alpha}, \bar{\beta}] = \langle \pi^{23}, \bar{\alpha} \otimes \bar{\beta} - \bar{\beta} \otimes \bar{\alpha} \rangle, \quad (8)$$

(the upper indices “23” of π stand for the contraction on the 2d and 3rd entry of the tensor product); \mathcal{G} acts on \mathcal{A} by

$$\rho(\varepsilon)(\bar{\alpha}) = \mathcal{L}_{\pi^\# \varepsilon}(\bar{\alpha}) - \langle \pi^{23}, \mathcal{D}_H \varepsilon \otimes \bar{\alpha} \rangle, \quad \text{for } \mathcal{D}_H \varepsilon = d\varepsilon + \iota_{\pi^\# \varepsilon} H. \quad (9)$$

Proof. First we have to show that the semi-direct product $\mathcal{G} \in \mathcal{A}$ is well defined, namely that the operation given by (9) indeed defines an action in agreement with a somewhat artificial Lie bracket (8). One can prove this fact by direct computations in some local chart, but it is rather lengthy, so to simplify it we notice that the first Lie derivative part of (9) is itself a Lie algebra action. Secondly, one may notice that the operator \mathcal{D}_H , defined on 1-forms behaves nicely with respect to the bracket, i.e.

$$\mathcal{D}_H[\varepsilon^1, \varepsilon^2] = \mathcal{L}_{\pi^\# \varepsilon^1}(\mathcal{D}_H \varepsilon^2) - \mathcal{L}_{\pi^\# \varepsilon^2}(\mathcal{D}_H \varepsilon^1).$$

The remaining part of the proof is just application of these observations to compare the right- and left-hand-sides in the definition of a Lie algebra action.

Now we will construct explicitly the maps defining the exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow \tilde{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 0$$

We can notice, that the first term in the formula (7) for the vector field $\tilde{\varepsilon}$ corresponds precisely to the element $\varepsilon \in \mathcal{G}$, from it we can recover the bracket of 1-forms as the derived bracket of vector fields ($\frac{\partial}{\partial p_i}$ part). If we perform a change of coordinates in (7) we notice that α_{ij} doesn't transform as a tensor, but being corrected by subtracting $\varepsilon_{i,j}$ it produces the 2-tensor $\bar{\alpha} \in \mathcal{A}$ and the respective bracket. This defines the desired isomorphism. \square

Remark 3.3. *Let us also note, that in the computations of [1] for the untwisted case the second term is defined by the lift of the vector field from T^*M to TT^*M by the Lie-derivative $\tilde{\varepsilon} = \mathcal{L}_\varepsilon$. That is the Lie algebra considered there is \mathcal{G} itself. (We used a similar construction in the theorem 3.1). And this lift in fact gives the form of the correction necessary to recover the Lie-algebra \mathcal{A} in our case.*

Now we define the condition that will be responsible for gauge invariance of the sigma model. Let us consider a subalgebra $\mathcal{GT} \subset \tilde{\mathcal{G}}$ defined by

$$d\varepsilon + \iota_{\pi\#\varepsilon}H = 0, \quad \bar{\alpha}^A = 0, \quad (10)$$

where $\bar{\alpha}^A$ is the antisymmetrization of the tensor $\bar{\alpha}$. Restricted to this subalgebra the Lie bracket (5) simplifies to $[\varepsilon^1, \varepsilon^2] = d(\pi(\varepsilon^2, \varepsilon^1)) + \iota_{\pi\#\varepsilon^2}\iota_{\pi\#\varepsilon^1}H$ and the action (9) to $\rho(\varepsilon)(\bar{\alpha}) = \mathcal{L}_{\pi\#\varepsilon}(\bar{\alpha})$, which preserves the symmetric tensor property.

Remark 3.4. *To define gauge transformations from \mathcal{GT} using (2) the first equality in (10) is necessary since it just repeats (6) from the theorem 3.1. The second one is the description of the extension of the algebra of symmetries by gauge transformations that are trivial in the sense of [22].*

3.2. Twisted Poisson sigma model from gauging

Having defined the algebra \mathcal{GT} we are ready to formulate the “converse” statement to the theorem 3.1. Given a closed 3-form H on M we can view it as a superfunction on $\tilde{\mathcal{M}} = T[1]T^*[1]M$. We search for a \mathcal{GT} -equivariantly closed extension (in the generalized sense of section 2) of H , that is a 3-form (or better to say a degree 3 superfunction) \tilde{H} on $\tilde{\mathcal{M}}$ which is \tilde{Q} -closed, \mathcal{GT} -equivariant and starts with H . The following statement holds true:

Theorem 3.2.

1. *Consider the graded manifold $\mathcal{M} = T[1]T^*[1]M$, equipped with the Q -structure $\tilde{Q} = \tilde{Q}_{\Pi,H}$, governed by an H -twisted Poisson bivector Π , such that*

the pull-back of H to a dense set of orbits of Π is not vanishing identically. The \mathcal{GT} -equivariantly closed extension of a given 3-form H is determined uniquely by

$$\tilde{H} = \frac{1}{6}H_{ijk}dx^i dx^j dx^k + \frac{1}{2}H_{ijk}\pi^{k'k}dx^i dx^j p_{k'} + dp_i dx^i.$$

2. Being pulled-back by f^* this extension defines the integrand of the twisted Poisson sigma model functional.

Proof. Although the natural local coordinates one introduces on the fiber $\tilde{\mathcal{M}}$ of the extended target are $(x^i(0), p_i(1), dx^i(1), dp_i(2))$ ³ it is more convenient to perform the computations in the basis generated by \tilde{Q} as a differential, i.e. $(x^i(0), p_i(1), \tilde{Q}x^i(1), \tilde{Q}p_i(2))$ (see also the proposition 1.1). The following expression for the vector field \tilde{Q} makes this coordinate change explicit:

$$\begin{aligned} \tilde{Q} = & \left(dx^i + \pi^{i'i}p_{i'}\right) \frac{\partial}{\partial x^i} + \left(dp_i - \frac{1}{2}C_i^{jk}p_j p_k\right) \frac{\partial}{\partial p_i} + \\ & + \left(d(\pi^{i'i}p_{i'})\right) \frac{\partial}{\partial dx^i} + \left(d(-\frac{1}{2}C_i^{jk}p_j p_k)\right) \frac{\partial}{\partial dp_i} \end{aligned} \quad (11)$$

The most general degree 3 superfunction on $\tilde{\mathcal{M}}$ reads

$$\begin{aligned} \omega = & A_{ijk}\tilde{Q}x^i\tilde{Q}x^j\tilde{Q}x^k + B_{ij}^k\tilde{Q}x^i\tilde{Q}x^j p_k \\ & + K_i^{jk}\tilde{Q}x^i p_j p_k + D^{ijk}p_i p_j p_k + E_i^j\tilde{Q}x^i\tilde{Q}p_j + F^{ij}\tilde{Q}p_i p_j, \end{aligned}$$

where the coefficients are superfunctions of degree 0, i.e. they depend only on x . On this superfunction we impose the condition of being \mathcal{GT} -basic in the sense of definition 2.3. The condition of \tilde{Q} -closedness reduces ω to

$$\omega' = A_{ijk}\tilde{Q}x^i\tilde{Q}x^j\tilde{Q}x^k + E_{i,j}^k\tilde{Q}x^i\tilde{Q}x^j p_k + \frac{1}{2}F_{,i}^{jk}\tilde{Q}x^i p_j p_k + E_i^j\tilde{Q}x^i\tilde{Q}p_j + F^{ij}\tilde{Q}p_i p_j,$$

for $A_{[ijk,l]} = 0$ and $F^{(ij)} = 0$.

Let us compute the action of the vector field $\tilde{\varepsilon}$ on this superfunction. Using its value on the generators: $\tilde{\varepsilon}\tilde{Q}x^i = \pi^{i'i}\varepsilon_{i'}$, $\tilde{\varepsilon}p_i = \varepsilon_i$, and $\tilde{\varepsilon}\tilde{Q}p_i = (\alpha_{ij} + \varepsilon_{i,j})\tilde{Q}x^j - ((\alpha_{ij} + \varepsilon_{i,j})\pi^{j'j} - C_i^{jk}\varepsilon_j)p_k$, we obtain four types of

³The numbers in brackets denote the total degree of each coordinate.

terms proportional to $\tilde{Q}x^j\tilde{Q}x^k$, $\tilde{Q}x^jp_k$, p_jp_k , $\tilde{Q}p_i$. The fact that they should vanish for all $\tilde{\varepsilon} \in \mathcal{GT}$ gives respectively the following conditions:

$$\begin{aligned} 3A_{i[jk]}\pi^{i'i}\varepsilon_{i'} + E_{[j,k]}^{i'}\varepsilon_{i'} + E_j^{i'}(\varepsilon_{i',[k} + \alpha_{i'[k}) &= 0, \\ 2E_{[i,j]}^k\pi^{i'i}\varepsilon_{i'} - F_{,j}^{i'k}\varepsilon_{i'} + E_j^{i'}(\varepsilon_{i',k'} + \alpha_{i'k'})\pi^{k'k} + C_j^{i'k}\varepsilon_{i'} + F^{i'k}(\varepsilon_{i',j} + \alpha_{i'j}) &= 0, \\ \frac{1}{2}F_{,i}^{[jk]}\pi^{i'i}\varepsilon_{i'} - F^{i'[j}((\varepsilon_{i',k'} + \alpha_{i'k'})\pi^{k]k'} + C_{i'}^{k]k'}\varepsilon_{k'}) &= 0, \\ E_i^j\pi^{i'i}\varepsilon_{i'} + F^{ii'}\varepsilon_{i'} &= 0. \end{aligned}$$

Considering first the restrictions from the first two of these equations coming from arbitrary symmetric α 's and vanishing ε 's (they always belong to \mathcal{GT}), one concludes that $E_j^i = a \cdot \delta_j^i$ and $F^{ij} = a \cdot \pi^{ij}$ for an arbitrary constant a . For this solution the forth equation is satisfied automatically and the third one due to the twisted Jacobi identity.

Since the constructed superfunction is the extension of H the values of all the components A_{ijk} coincide with $\frac{1}{6}H_{ijk}$. It remains now to fix the relative prefactor a between A and the remaining terms. Due to the non-degeneracy condition that is asked in the theorem, the first equation necessarily produces a restriction on a which is satisfied precisely for $a = 1$, that is

$$\tilde{H} = \frac{1}{6}H_{ijk}\tilde{Q}x^i\tilde{Q}x^j\tilde{Q}x^k + \frac{1}{2}\pi_{,i}^{jk}\tilde{Q}x^ip_jp_k + \tilde{Q}x^i\tilde{Q}p_i + \pi^{ij}\tilde{Q}p_ip_j,$$

Rewriting the expression in the usual basis of $T[1]T^*[1]M$ completes the proof of the first point of the theorem.

To verify the second point one needs simply to compute the pull-back by f^* of the resulting superfunction. Let us note here that thanks to the choice of the basis on the fiber generated by \tilde{Q} the computation is very explicit using only the proposition (1.1), this yields

$$\begin{aligned} S[X, A] = \int_{\Sigma_3} f^*(\tilde{H}) &= \int_{\Sigma_3} \frac{1}{6}H_{ijk}dX^i dX^j dX^k + \\ &+ \frac{1}{2}\pi_{,i}^{jk}dX^i A_j A_k + dX^i dA_i + \pi^{ij}dA_i A_j. \end{aligned}$$

To recover precisely the functional of the twisted Poisson sigma model one needs to perform the usual partial integration of the result using Stokes' theorem. \square

Remark 3.5. *If $H = 0$, i.e. in the “untwisted” case, one obtains the functional of the Poisson sigma model up to a physically irrelevant constant prefactor. One can also slightly generalize the picture for $H \neq 0$, relaxing the condition that \tilde{H} starts with H . Then in the resulting formula for \tilde{H} one can modify $A = H$ by a term H' such that it is still closed and the contraction of H' with the anchor map $\pi^\#$ vanishes.*

Remark 3.6. *We should also note, that in the pure (untwisted) Poisson case one can ignore the 2-tensor contribution of $\bar{\alpha}$ in (7) and restrict oneself to the analysis of $T^*[1]M$ as the manifold where the generators of the gauge transformations are defined (i.e. completely reverse the statement of the proposition 3.1). However in the twisted case one does have to consider the 2-tensor contribution, as for non-zero H (10) could be very restrictive on ε , that is the Lie algebra \mathcal{G} would be too small to define uniquely \tilde{H} .*

Remark 3.7. *The condition of non-degeneracy of the pull-back of H to the orbits of Π seems restrictive from the first sight. But actually this is a sufficient condition that permits to guarantee a relation between the two parts of \tilde{H} not describing explicitly the subset of \mathcal{GT} generated by ε ’s for vanishing α ’s. We know however some examples when the condition is not fulfilled but the extension is still unique – it would be interesting to study this phenomenon in details.*

Remark 3.8. *The statement similar to the above theorem can be proved also for the Dirac sigma model (DSM) ([23], [24]), one can even attempt to generalize the construction to a Lie algebroid structure on the target. However the proof that we have presented here already encounters the key ideas and difficulties and in contrast to the DSM we do not have to use any auxiliary structure.*

4. Equivariant cohomology of Courant algebroids

In this short section let us turn to a completely mathematical application of equivariant Q -cohomology. We will propose a definition of equivariant cohomology of Courant algebroids. We do it not with some computational purpose but more to show that the framework of the section 2 is indeed large.

Definition 4.1. *A Courant algebroid is a vector bundle $E \rightarrow M$ equipped with the following operations: a symmetric non-degenerate pairing $\langle \cdot, \cdot \rangle$*

on E , an \mathbb{R} -bilinear bracket $[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$ on sections of E , and an anchor ρ which is a bundle map $\rho : E \rightarrow TM$, satisfying the axioms:

$$\begin{aligned}\rho(\varphi) \langle \psi, \psi \rangle &= 2 \langle [\varphi, \psi], \psi \rangle, \\ [\varphi, [\psi_1, \psi_2]] &= [[\varphi, \psi_1], \psi_2] + [\psi_1, [\varphi, \psi_2]], \\ 2[\varphi, \varphi] &= \rho^*(d \langle \varphi, \varphi \rangle),\end{aligned}$$

where $\rho^* : T^*M \rightarrow E$ (identifying E and E^* by $\langle \cdot, \cdot \rangle$).

We will restrict the analysis to a *twisted exact Courant algebroid structure* on $E = TM \oplus T^*M$, governed by a closed 3-form H on M (for a review on the subject see for example [25]). The symmetric pairing is given by $\langle v \oplus \eta, v' \oplus \eta' \rangle = \eta(v') + \eta'(v)$, the anchor $\rho(v \oplus \eta) = v$ and the bracket is the H -twisted Courant–Dorfman bracket

$$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\text{Lie}} \oplus (\mathcal{L}_v \eta' - \iota_{v'} d\eta + \iota_v \iota_{v'} H). \quad (12)$$

According to [10] Courant algebroids are in bijection with degree 2 symplectic Q -manifolds, that is one can construct a Q structure associated to E . To do this we consider the graded manifold $\mathcal{M} = T^*[2]T[1]M$ and choose local coordinates on it: $(p_i(1), \psi_i(2), \theta^i(1), x^i(0))$. On $T^*[2]T[1]M$ there is a canonical (degree -2) Poisson bracket. The Q -structure can thus be constructed as a hamiltonian vector field $Q_{CA} = \{Q, \cdot\}$, where $Q = \psi_i \theta^i + \frac{1}{6} H_{ijk} \theta^i \theta^j \theta^k$.

For the degree -1 vector fields let us consider degree $+1$ super functions on \mathcal{M} and hamiltonian vector fields associated to them. The most general degree $+1$ function is of the form $\epsilon = \alpha_i \theta^i + v^i p_i$ for α_i, v^i – some smooth functions of x . Considering the Q_{CA} derived bracket of the vector fields $\epsilon := \{\epsilon, \cdot\}$ one recovers the structure similar to (12).

It is now natural to give the following definition:

Definition 4.2. *The equivariant cohomology of a Courant algebroid E is described by the definitions 2.1, 2.2, 2.3 with the Q -structure Q_{CA} and the algebra $\mathcal{G} = \{\epsilon\}$ or any subalgebra of it.*

Remark 4.1. *Let us note that this definition already allows to treat the action of a huge algebra, including for example all vector fields on M . But the construction is not limited to it, namely we are not forced to restrict the vector fields ϵ to hamiltonian ones, instead we can consider all the degree -1 vector fields on \mathcal{M} . If even this is not enough we can perform the lift to $T[1]\mathcal{M}$ preserving the pattern of the definition.*

5. Conclusion/discussions

In this paper we have observed on the example of the concept of equivariant Q -cohomology how the problem from theoretical physics can motivate the definition of a purely mathematical structure which can then again be used to analyze physical problems. Let us conclude by giving some small remarks and open questions inspired by the above presentation.

First, concerning the twisted Poisson sigma model: we have obtained the condition (6) describing the algebra of its symmetries (theorem 3.1). It actually has a nice geometric interpretation: in the case of a non-degenerate closed 3-form H it is nothing but saying that $\pi^\# \varepsilon$ is a 2-hamiltonian vector field with respect to a 2-symplectic form H (cf. [26]). This is even more visible in the analogous statement for the Dirac sigma model ([24]). Moreover if one considers other dimensions of Σ the condition transforms to a system of partial differential equations including for example the stationary Lamb equation from hydrodynamics ([27]). One is thus tempted to apply the information from these mathematical structures to study the symmetries of physical theories or conversely profit from knowledge of equivalent gauge theories to describe for example local geometry of twisted Poisson manifolds or Dirac structures.

Second, concerning the gauging problem: we have described a constructive procedure to produce gauge invariant functionals. We expect this approach to be useful to study some existing theories, like Lie algebroid Yang-Mills ([28]) or gauge theories on transitive Lie algebroids ([29]), as well as to produce new ones (like in the proof of the theorem 3.2), since there is no a-priori restriction neither on the dimension of the manifolds involved, nor on the gauge group. We have seen that the symmetry algebra can be infinite dimensional, like the one defined by (6), an interesting question to ask here is how to characterize the algebras for which gauging is not obstructed ([30], [31]).

Third, inspired by the construction of the section 4 we expect the similar procedure to be related to existing definitions of equivariant cohomology of Lie algebroids ([32], [33], [34]).

Acknowledgements. I would like to thank Thomas Strobl for constant attention towards this work. I also appreciate inspiring discussions with Alexei Kotov and Vladimir Roubtsov. This paper being an extended version of the poster presented at the FDIS13 conference and the talk delivered

during the CRM research program “Geometry and dynamics of integrable systems”, I would like to thank the organizers of these events for inviting me, as well as the participants for valuable comments and questions.

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